QCD in a box and Adiabatic continuity

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Based on the following works:
1710.08923, 1711.10487 (with Yuta Kikuchi, Tatsuhiro Misumi, Norisuke Sakai)
2201.06166 (with Mithat Ünsal)
2402.04320, 2405.12402 (with Yui Hayashi) + on-going (with Hayashi, Ünsal)

Quantum Chromo Dynamics (QCD)

Fundamental theory of strong interaction:

- A_μ : gluons [SU(N$_c$) gauge field]
- q : quarks [Dirac fermion in the fund. rep.]

Lagrangian is very simple:

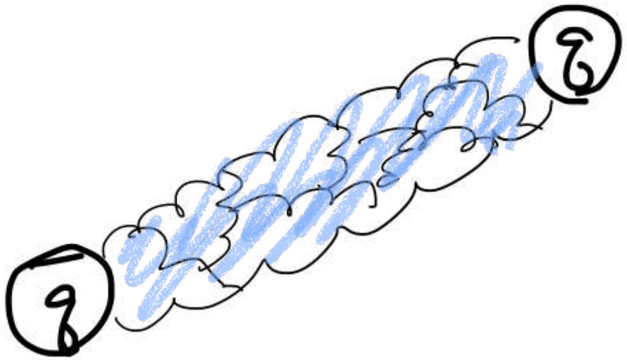
$$\mathcal{L} = \frac{1}{g^2} \mathrm{tr}\left(F_{\mu\nu} F_{\mu\nu}\right) + \bar{q}\left(\not{D} + m\right) q .$$

All interesting phenomena of nuclear/hadron physics should come out of it!

① Confinement

quarks & gluons can never be observed as an isolated particle :

quarks are "confined" in a color-singlet compact bound state (hadron).



Even if we try to separate two test quarks, they are connected by a color flux tube and cannot be separated with a finite energy.

② Chiral symmetry & its spontaneous breaking

In the massless limit ($m=0$), the left- & right-handed quarks decouple:

$$\bar{q} \not{D} q = \bar{q}_L \not{D} q_L + \bar{q}_R \not{D} q_R$$

$\Rightarrow$ Chiral symmetry $SU(N_f)_L \times SU(N_f)_R \times U(1)_{diag.}$

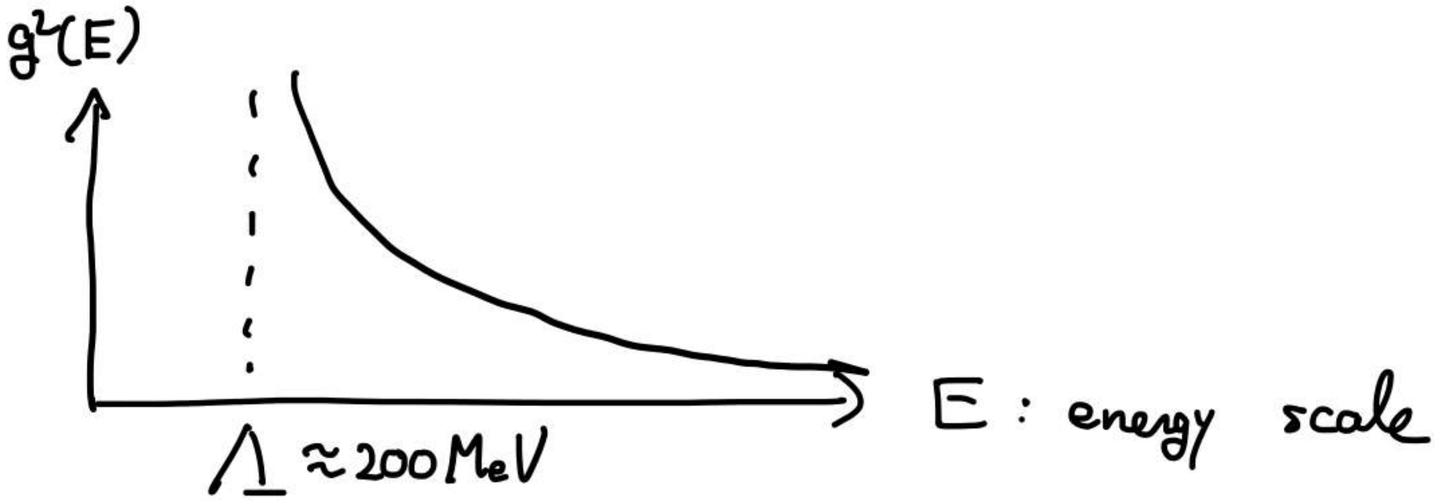
In the QCD vacuum, this is spontaneously broken:

$$\langle \bar{q}_L q_R \rangle = - \# \Lambda^3 \neq 0.$$

$\Rightarrow$ massless pions as Nambu – Goldstone modes.

Both ① confinement and ② chiral SSB are nonperturbative phenomena.

- Low-energy QCD is strongly coupled due to asymptotic freedom:



$\Rightarrow$ We would like to get a new perspective of these phenomena:

Generalized Symmetry & Sym.-Twisted boundary condition

Generalized Symmetry

Conventional perspective :

Invariance of $\mathcal{L}(\phi) \Rightarrow$ Noether current J^μ w/ $\partial_\mu J^\mu \overset{\text{on-shell}}{=} 0$.

$\overset{\text{quantize}}{\Rightarrow}$ Ward-Takahashi identity $\partial_\mu \langle J^\mu(x)\, \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \sum_i \delta(x - x_i) \langle \mathcal{O}_1 \cdots \delta\mathcal{O}_i \cdots \mathcal{O}_n \rangle$.

↳ Various theorems

(selection rule, Nambu-Goldstone thm, anomaly matching ---)

Modern View : (Gaiotto, Kapustin, Seiberg, Willett '14, ----)

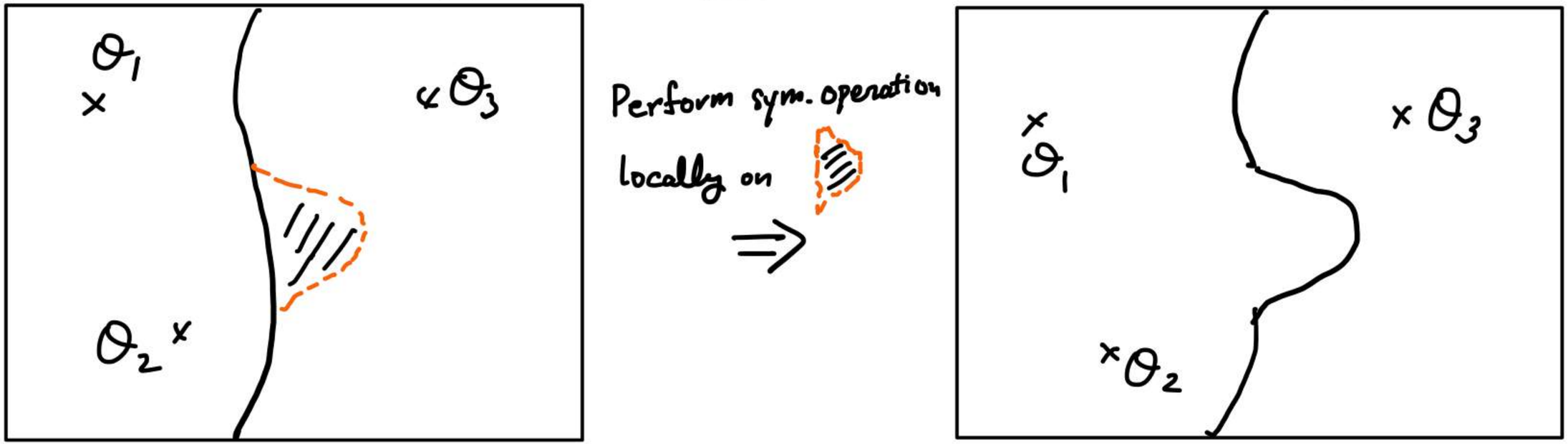
Extract the most essential ingredient of the WT identity.

$\Rightarrow$ **Symmetry = Topological operator**

Topological operator

Idea: For local QFTs, the symmetry operation can be performed locally!

$U(\Sigma)$: sym. defect on a surface Σ



$$\langle U(\Sigma)\, \theta_1 \cdots \theta_n \rangle = \langle U(\Sigma')\, \theta_1 \cdots \theta_n \rangle .$$

Generalized Symmetry = Data of Topological Operators

(1) Ordinary Symmetry is a special case s.t.

- $U_g(\Sigma)$: top. operators obey the group-like fusion
- They act on local (point-like) operators

(since 2014)

Higher-form/group symmetry

- Transformation on extended operators

e.g. Center sym. of YM theory

→ Next page

(since 2017)

Non-invertible symmetry

- More general fusion rule.

e.g. Kramers-Wannier duality symmetry for Ising CFT.

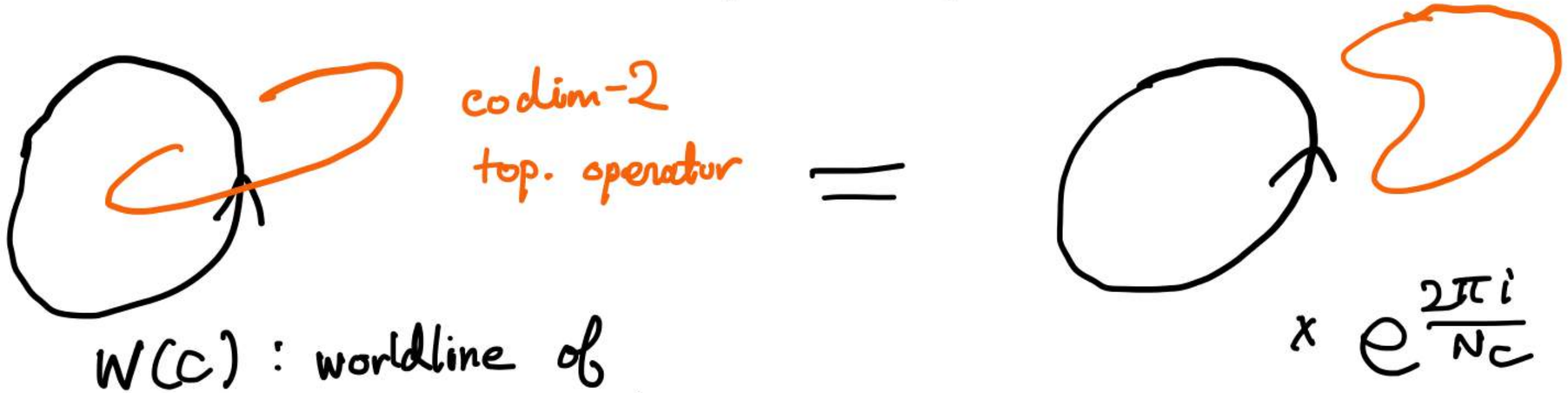
Yang-Mills theory & 1-form symmetry

Massive quark limit $m \to \infty$: Ideal limit to discuss quark confinement.

$\Rightarrow$ $SU(N_c)$ YM $\quad \mathcal{L} = \frac{1}{g^2} \mathrm{tr}(F_{\mu\nu}^2)$.

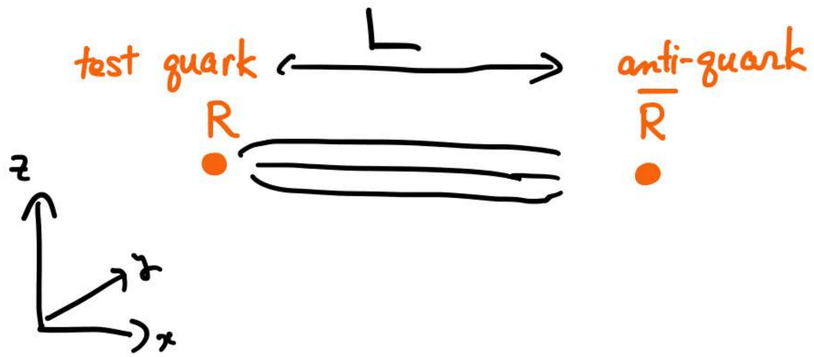
This theory has a <u>codim.-2</u> <u>topological</u> operator :

$\mathbb{Z}_{N_c}$ 1-form symmetry



$W(C)$: worldline of test quark

Confinement & Wilson loop



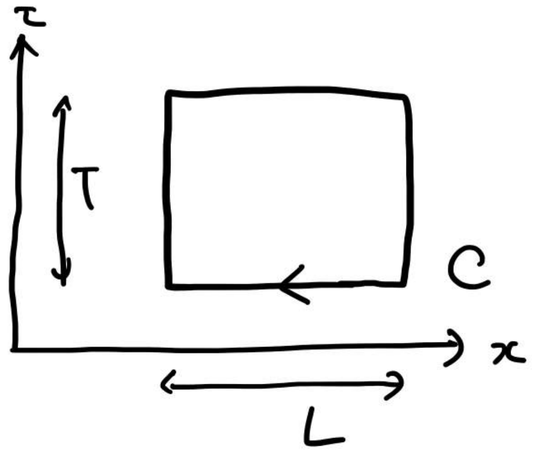
Interquark potential :

$$V_{R\bar{R}}(L) \simeq \begin{cases} \sigma_R L & \text{(confinement)} \\ \text{const.} & \text{(Higgs)} \end{cases}$$

Wilson loop : $W_R(C) = \mathrm{tr}_R \left[\mathcal{P} \exp\left(\oint_C a \right) \right]$.

$$\langle W_R(C) \rangle \simeq \exp\left(-T \cdot V_{R\bar{R}}(L) \right)$$

$$\simeq \begin{cases} \exp(-\sigma_R T L) & \text{(Area law)} \\ \exp(-\text{const.}\,(T+L)) & \text{(Perimeter law)}. \end{cases}$$



This is an order parameter for the $\mathbb{Z}_N$ 1-form symmetry, or center symmetry.

Background gauge field for $\mathbb{Z}_{N_c}^{(1)}$

Gauge field $\approx$ Aharonov-Bohm phase during the motion of particles, strings, etc.

$=$ Network of symmetry defects.

Network of codim-2 defects ($\mathbb{Z}_{N_c}^{(1)}$ generator)

$= \mathbb{Z}_{N_c}$ 2-form gauge field B

$Z[B]$ ← partition func. w/ the B flux ($=$ 't Hooft twist.)

$$Z[B] = \begin{cases} 0 & (\text{if } \mathbb{Z}_{N_c}^{(1)} \text{ is spontaneously broken}) \\ Z[0] \text{ up to a phase factor} & (\text{if unbroken}). \end{cases}$$

Symmetry of QCD

At finit quark masses $m < \infty$, $\mathbb{Z}_{N_c}^{(1)}$ is explicitly broken.

However, it has a subtle effect for the global symmetry:

$$\frac{SU(N_f)_{\text{diag}} \times U(1)_{\text{diag}}}{\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}}.$$

$\Rightarrow$ When activating its background gauge field,

it contains not only the conventional flavor gauge field

but also B_c : $\mathbb{Z}_{N_c}$ 2-form gauge field.

[Kikuchi, YT '17, Shimizu, Yonekura '17, Gaiotto, Komargodski, Seiberg '17, ---]

Refined symmetry data : New symmetry constraint

If we activate the background flavor gauge field including B_c, B_f,
the QCD partition function has a nontrivial responce under chiral symmetry:

$$Z_{QCD}[A, B_c, B_f] \xrightarrow{\text{discrete chiral operation at } m=0} e^{i\frac{1}{N_f}\int (N_c dA_V + 2\pi B_c)\cup B_f} \times Z_{QCD}.$$

$\Downarrow$ Persistent order for the suitably chosen BCs.

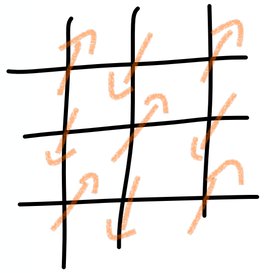
$\mathbb{Z}_N$ - twisted QCD phase diagram (When $N_c = N_f = N$)

[Kikuchi, Misumi, YT, Sakai '17; cf. Shimizu, Yonekura '17]

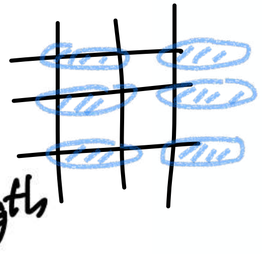
(Analogous to the Néel - VBS transition by Sachdev et. al.) → Next page

Similarity with (2+1)D antiferromagnet

AF Heisenberg + four-spin interaction (Sachdev, Senthil, ---)



Néel state — Valence-bond solid → 4-spin interaction strength



Direct transition from an $SO(3)_{spin}$-broken state to a lattice-sym. broken state.

Direct transition is a consequence of LSM-type argument (Senthil et al. '04)
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Discrete anomaly matching (Komargodski, Shamon, Thorngren, Zhou '17)

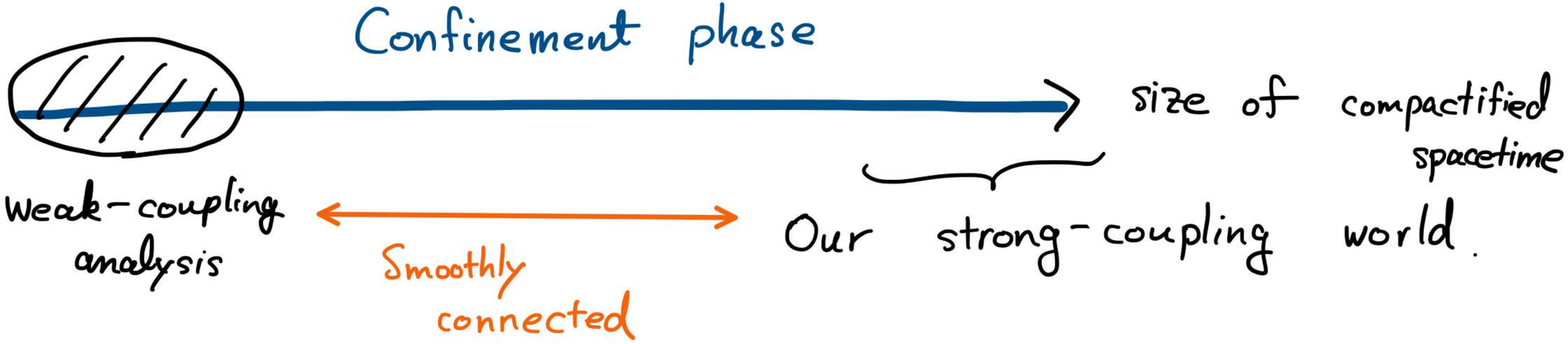
($\mathbb{Z}_N$-twisted QCD and/or QCD at Roberge-Weiss point

chiral broken — deconfined → Inverse size of S^1 ($\simeq$ temperature)

direct transition (most likely))

Adiabatic Continuity to Weak-Couplings

We would like to improve our understandings about microscopic mechanism of confinement and related phenomena.



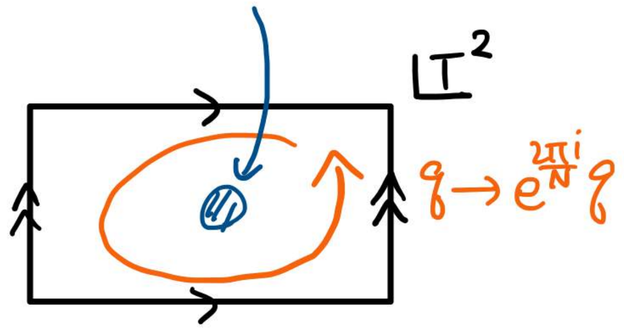
cf. · Twisted Eguchi-Kawai model (Gonzalez-Arroyo, Okawa)

· Double-trace deformed YM or YM theory w/ adjoint fermions } on $\mathbb{R}^3 \times S^1$ (Ünsal, ... 2007–)

Conjecture (YT, Ünsal, '22)

YM, QCD on $\mathbb{R}^2 \times T^2$

w/ 't Hooft flux



Weak-coupling description

$\Longleftrightarrow$

Adiabatic Continuity

YM, QCD on $\mathbb{R}^4$

Strong-couplings

via Center Vortices

(Related works:
van Baal '84 (thesis), Yamazaki, Yonekura '17, Cox, Poppitz, Wandler '21 on $T^3 \times \mathbb{R}$)

Connecting dynamics from 4D to 2D

- YM/QCD on $\mathbb{R}^2 \times T^2 \Rightarrow$ 2D effective theory on $\mathbb{R}^2$

We want 2D dynamics to reproduce 4D confinement as much as possible.

Idea : Maintain the information of the 't Hooft anomaly !

4D YM : $Z^{(4D)}_{\theta+2\pi}[B] = e^{\frac{2\pi i}{N}\int B \cup B} \, Z^{(4D)}_{\theta}[B]$ (Gaiotto, Kapustin, Komargodski, Seiberg '17)

T^2- compactification

't Hooft twist.

$$B = B_{2d} + A\,\frac{dx_3}{L} + \tilde{A}\,\frac{dx_4}{L} + p\,\frac{dx_3 \wedge dx_4}{L^2}$$

$$Z^{(2D)}_{\theta+2\pi}[B_{2d}, A, \tilde{A}] = e^{\frac{2\pi i}{N}\int (p B_{2d} - A \cup \tilde{A})} \, Z^{(2D)}_{\theta}[B_{2d}, A, \tilde{A}]$$

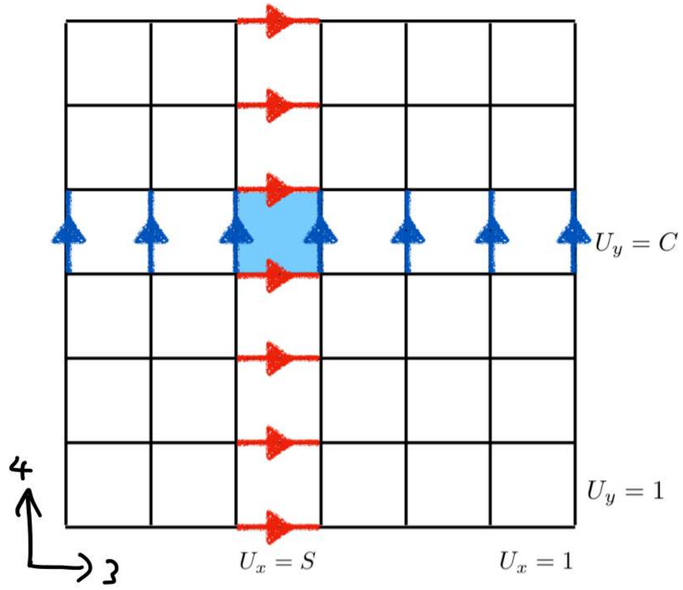
Only if we suitably choose p ($\gcd(N,p)=1$), the 4D anomaly is maintained in the 2D effective theory.

't Hooft flux & Classical vacuum

Lattice action

$$S_W[U_\ell, B] = -\frac{1}{g^2}\sum_P \left(e^{-iB_P}\, \mathrm{tr}[U_P] + e^{iB_P}\, \mathrm{tr}[U_P^\dagger] \right).$$

$$B_P = \begin{cases} \frac{2\pi}{N} & \text{(for the plaquette indicated with light blue)} \\ 0 & \text{(otherwise)} \end{cases}$$



We can minimize this action by setting

$$U_\ell = \begin{cases} S = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix} \\ C = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & \omega^{N-1} \end{pmatrix} \\ \mathbb{1} \end{cases}$$

[* Any classical minimum is gauge equivalent to this one]

$$\Rightarrow \quad P_3 = S \,, \quad P_4 = C \,.$$

This configuration completely preserves $\mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$.

2d Wilson loop & Center-vortex fractional instantons

P_3, P_4 : Wilson loops in compactified T^2 directions $\Rightarrow$ Confined

What about $W(C)$ in the $\mathbb{R}^2$ direction?

$\Leftarrow$ Vortex solitons are important!

- Self-dual YM solution

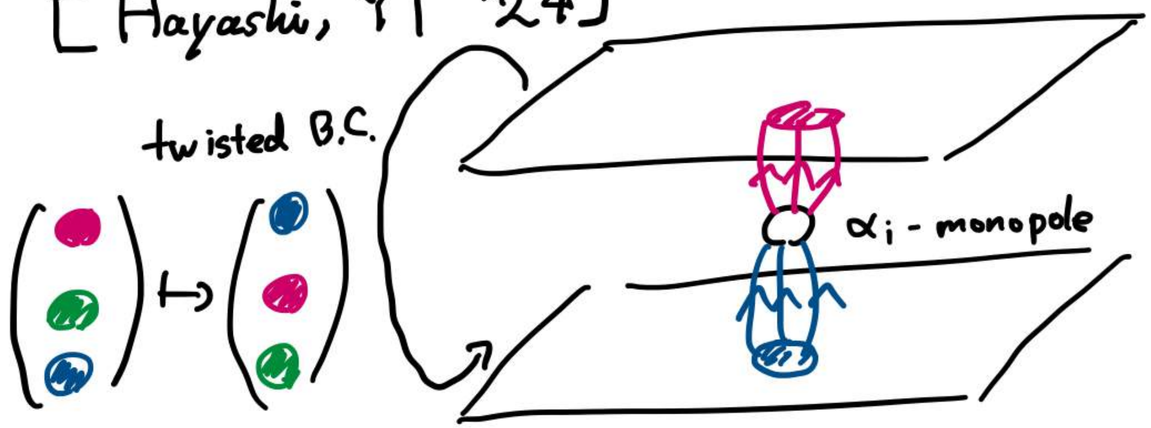
$$F = *F$$

- $Q_{top} = \frac{1}{N}$

[Numerically found in González-Arroyo, Montero '97]

- Fraction of the doubly-periodic instanton obtained by Kraan-van Baal-Lee-Yi caloron.

[Hayashi, YT '24]



Partition funcion on $M_2 \times T^2$ & θ-dependence

$M_2 \to \mathbb{R}^2$

To make the computation well-defined, we compactify $\mathbb{R}^2$ to some closed 2-manifold M_2.

Using the 1-loop vertex of the center vortex

$$K \cdot e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}},$$

we have

$$\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N} k(n-\bar{n})} = \delta_{n-\bar{n} \in N\mathbb{Z}}$$

$$Z(\theta) = \sum_{n,\bar{n} \geq 0} \frac{\delta_{n-\bar{n}\in N\mathbb{Z}}}{n!\,\bar{n}!} \left(V \cdot K e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}} \right)^n \left(V \cdot K e^{-\frac{8\pi^2}{g^2 N} - i\frac{\theta}{N}} \right)^{\bar{n}}$$

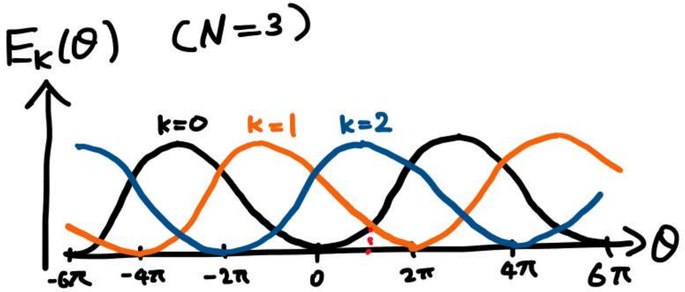
(vortex: $e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}}$; anti-vortex: $e^{-\frac{8\pi^2}{g^2 N} - i\frac{\theta}{N}}$)

$$= \sum_{k=0}^{N-1} \exp\left[-V \left(-2K e^{-\frac{8\pi^2}{g^2 N}} \cos\left(\frac{\theta - 2\pi k}{N} \right) \right) \right]$$

$E_k(\theta)$: Ground-state energy densities

$\Rightarrow$

- N-branch structure of ground states.
- Each branch has a fractional θ-dependence.



Physical Meaning of the branch label $k \in \mathbb{Z}_N$

① Distinction as Symmetry-Protected Topological (SPT) states

When 4d $\mathbb{Z}_N^{(1)}$ is unbroken,

$$Z[B] \overset{\text{low energy}}{\sim} \exp\left(i \int \frac{N k}{4\pi} B \wedge B\right).$$

[cf. Gaiotto, Kapustin, Komargodski, Seiberg '17]

② Distinction based on Wilson-'t Hooft classification

't Hooft ('79): Area law for $W(C)$ is not enough to classify confinement states.
Use also $H(C,\Sigma)$.

Screened line operators are generated by $H(C,\Sigma) \cdot W^k(C)$.

(ex $SU(2)$)

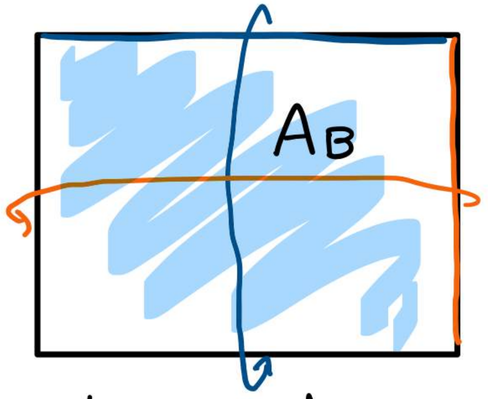
	$W(C)$	$H(C,\Sigma)$	$W(C)H(C,\Sigma)$
Higgs	**Perimeter**	Area	Area
Confinement $\theta \sim 0$	Area	**Perimeter**	Area
Confinement $\theta \sim 2\pi$	Area	Area	**Perimeter**.

[Equivalence ① $\Leftrightarrow$ ②: Nguyen, YT, Ünsal 2306.02485]

$U(1)_B$ monopole flux & 't Hooft flux on T^2

Fundamental quarks explicitly violates $\mathbb{Z}_N^{(1)}$ $\Rightarrow$ No 't Hooft B.C., naively.

Use $U(1)_B = \frac{U(1)_q}{\mathbb{Z}_N}$ monopole flux.



Cocycle condition

$$\begin{cases} \psi(L, x_4) = g_3^\dagger(x_4)\, e^{-i\frac{\phi_3(x_4)}{N}}\, \psi(0, x_4) \\ \psi(x_3, L) = \underbrace{g_4^\dagger(x_3)}_{\text{color-transition functions}}\, \underbrace{e^{-i\frac{\phi_4(x_3)}{N}}}_{U(1)_q\text{-transition functions}}\, \psi(x_3, 0) \end{cases}$$

$$g_3^\dagger(L)\, g_4^\dagger(0)\, e^{-i\frac{1}{N}(\phi_3(L)+\phi_4(0))} = g_4^\dagger(L)\, g_3^\dagger(0)\, e^{-i\frac{1}{N}(\phi_4(L)+\phi_3(0))}.$$

$U(1)_B = U(1)_q / \mathbb{Z}_N$ monopole flux

$$2\pi = \int_{T^2} dA_B = (\phi_3(L) - \phi_3(0)) - (\phi_4(L) - \phi_4(0))$$

$$\Rightarrow \quad g_3^\dagger(L)\, g_4^\dagger(0) = g_4^\dagger(L)\, g_3^\dagger(0)\, \boxed{e^{\frac{2\pi i}{N}}}$$

't Hooft flux !!

2d Effective Lagrangian on $\mathbb{R}^2 \times T^2$ w/ baryon - 't Hooft flux

2d $\mathbb{Z}_N$ Schwinger model

4d QCD $\overset{\text{classical \& Perturbative}}{\Longrightarrow}$

- Gauge : Gapped gluons & Center vortices
- Quark : 2d Dirac fermion as 4d Dirac zero modes w/ $\int_{T^2} dA_B = 2\pi$.

$\Downarrow$ Bosonization & Semiclassical Analysis

U : 2d $U(N_f)$-valued field (π, K, η & η') [YT, Ünsal 2201.06166; Hayashi, YT 2402.02430]

$$\mathcal{L}_{\text{2d effective}} = \frac{1}{8\pi} \mathrm{tr}(\partial_\mu U^\dagger \partial_\mu U) - \mathrm{tr}[M U + \text{c.c.}]$$
$$+ i \frac{1}{12\pi} \mathrm{tr}\left[(U^\dagger dU)^3\right]$$
$$- e^{-\frac{8\pi^2}{g^2 N}} \left(e^{-\frac{i\theta}{N}} (\det U)^{1/N} + \text{c.c.} \right)$$

[$\Leftarrow$ If we neglect massive η' part, this is consistent w/ 4d chiral Lagrangian on $\mathbb{R}^2 \times T^2$ w/ $\int_{T^2} dA_B = 2\pi$.]

Revisiting $U(1)_A$ problem

$U(1)_A$: η' (i.e. $e^{i\eta'} = \det U$) is too massive according to SSB of the chiral symmetry $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$.

$\Rightarrow$ $U(1)_A$ is not a symmetry of quantum theory !

η' gets the mass even if $M_{quark} = 0$.

Previous proposals

- Kobayashi-Maskawa-'t Hooft : $-\cos(\eta' - \theta)$
- Witten-Veneziano's large-N : $\eta'^2 + O\left(\frac{\eta'^4}{N^2}\right)$

large-N

$\frac{1}{N}$ fractionalize

2d center-vortex theory : $-\cos\left(\frac{\eta' - \theta}{N}\right)$

How does $\frac{1}{N}$ appear? What is its fermionic picture?

$\mathbb{R}^2 \times T^2$
w/ 't Hooft flux
$\Longrightarrow$

4d instanton = N independant
center vortices

$$e^{-\frac{8\pi^2}{g^2}} \cdot e^{-i\theta} \det U(x)$$

$$\prod_{i=1}^{N} e^{-\frac{8\pi^2}{g^2 N}} e^{-i\frac{\theta}{N}} \left(\det U(x_i)\right)^{1/N}$$

each term behaves almost independently

Periodicity of η'

Recall that we introduced η' by

$$e^{i\eta'} = \det U.$$

$\Rightarrow \eta' \sim \eta' + 2\pi$, which does not fit $-\cos\left(\frac{\eta'-\theta}{N}\right)$.

[Same problem occurs for the Witten–Veneziano formula]

Solution Reinstate the YM confinement vacua labelled by $k \sim k+N$.

$$-e^{-\frac{8\pi^2}{g^2N}}\cos\left(\frac{\eta'-\theta}{N}\right) \Longrightarrow -e^{-\frac{8\pi^2}{g^2N}}\cos\left(\frac{\eta'-\theta+2\pi k}{N}\right).$$

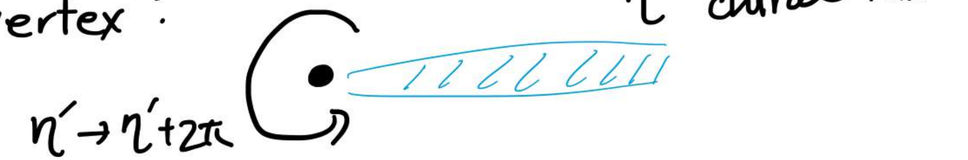
The correct periodicity for η' involves the shift of k : (Hayashi, YT 2402.04320)

$$(\eta'+2\pi, k) \sim (\eta', k+1).$$

$\Rightarrow \eta' \sim \eta' + 2\pi N$ by "eating" the k index.

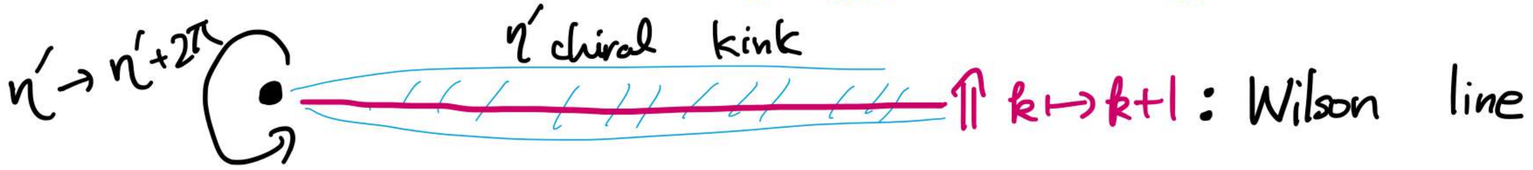
η' – Vortex

Usual KMT vertex :



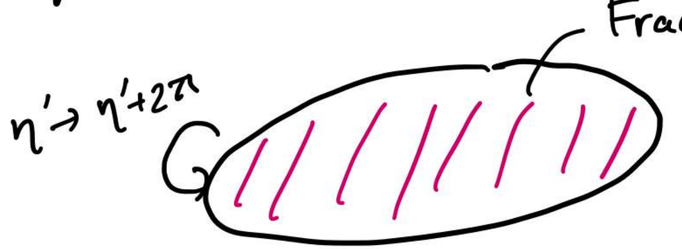
$\frac{1}{N}$ fractionalized vertex : $V(\eta', k) = -\cos\left(\frac{\eta' + \theta - 2\pi k}{N}\right)$ does not come back even if we shift $\eta' \to \eta' + 2\pi$.

k should also jump!



$\Rightarrow$ In $\mathbb{R}^2 \times T^2_{\text{twist}}$, **$\eta'$ – vortex = quark.**

This corresponds to the 2d reduction of the "pancake baryon" construction. (Komargodski 1812.09253)



QCD on $\mathbb{R}^2 \times T^2$ w/ Baryon- & 't Hooft flux

- Weakly-coupled if $NL\Lambda \ll 1$

  $\rightsquigarrow$ We can solve QCD analytically!

  $\Downarrow$

  2d $\mathbb{Z}_N$ – Schwinger model

- η' – potential involves the discrete label $k \in \mathbb{Z}_N$: [YT, Ünsal '22; Hayashi, YT '24]

  $$-\cancel{\cos(\eta' + \theta)} \quad \Longrightarrow \quad -\cos\left(\frac{\eta' + \theta + 2\pi k}{N}\right)$$

- η' – vortex = Boundary of $U(1)_N$ Chern-Simons quantum liquid

Summary

(I) Generalization of Symmetry

$\Rightarrow$ Refined tool to study / constrain strongly coupled dynamics

(II) QCD on $\mathbb{R}^3 \times S^1_{\text{twisted}}$ enjoys discrete anomalies for suitably chosen twists.

- Confinement $\Rightarrow$ Chiral SSB
- Interplay w/ quantum anti-ferromagnets ??

(III) QCD on $\mathbb{R}^2 \times T^2_{\text{Baryon flux}}$ gives a weakly-coupled setup to concretely study 4D confinement vacua.